SHORT EQUATIONAL BASES FOR MV-ALGEBRAS, COMMUTATIVE BCK-ALGEBRAS AND ŁBCK-ALGEBRAS

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Abstract. We show that the variety of MV-algebras is 2-based and we offer elegant 2-bases for the varieties of commutative BCK-algebras and ŁBCK-algebras.

1. Introduction

In this paper, we offer short equational bases for three varieties of algebras closely related to logic, namely MV-algebras, commutative BCK-algebras and ŁBCK-algebras. This work is in the same spirit of many other papers in which the general aim is provide simple systems of identities for various structures, where simplicity is roughly measured by the number of identities, or the number of symbols used, or the length of the identities or combinations of these criteria. We refer the interested reader to the extensive bibliography of [1].

MV-algebras, which are algebraic counterparts of Łukasiewicz logic, are algebras $(A, \oplus, \neg, 0)$ of type $\langle 2, 1, 0\rangle$ satisfying the following identities

$$(x \oplus y) \oplus z = x \oplus (y \oplus z) \tag{A1}$$

$$x \oplus y = y \oplus x \tag{A2}$$

$$x \oplus 0 = x \tag{A3}$$

$$\neg\neg x = x \tag{A4}$$

$$x \oplus \neg 0 = \neg 0 \tag{A5}$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \tag{A6}$$

The preceding definition comes from [3]. Other definitions exist using more operations, but these are definable in terms of 0, $\neg$ and $\oplus$. Since it turns out that $0 = \neg(\neg x \oplus x)$, the constant 0 can be removed from the signature of an MV-algebra, and the identities above can be appropriately modified. Thus MV-algebras can also be viewed as algebras of type $\langle 2, 1\rangle$.

Cattaneo and Lombardo gave a system of five independent axioms in terms of 0, $\neg$ and $\oplus$ for MV-algebras [2]. Our first main result is that the variety of MV-algebras (as algebras of type $\langle 2, 1\rangle$) is 2-based.

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Theorem 1.1. *The following identities form a basis for the variety of MV-algebras:*

$$\neg(x \oplus (\neg x \oplus y)) \oplus z = z \tag{M1}$$

$$\neg(\neg(x \oplus y) \oplus \neg(z \oplus u)) \oplus \neg(z \oplus (u \oplus \neg(x \oplus (y \oplus (z \oplus u))))) = x \oplus y\,. \tag{M2}$$

In an MV-algebra $(A, \oplus, \neg, 0)$, set $x \to y = \neg x \oplus y$ and $1 = \neg 0$. Then $(A, \to, 1, 0)$ is a bounded, commutative BCK-algebra. An algebra $(A, \to, 1)$ of type $\langle 2, 0\rangle$ is a *commutative BCK-algebra* if it satisfies the identities

$$(x \to y) \to y = (y \to x) \to x \tag{B1}$$

$$x \to (y \to z) = y \to (x \to z) \tag{B2}$$

$$x \to x = 1 \tag{B3}$$

$$1 \to x = x\,. \tag{B4}$$

This basis is due to H. Yutani [11]; we will not need the larger quasivariety of BCK-algebras in this paper. The constant 1 can be eliminated so that a commutative BCK-algebra can be viewed as an algebra of type $\langle 2\rangle$ by replacing (B3) with $x \to x = y \to y$ and replacing (B4) with $(x \to x) \to y = y$.

It is known that the variety of commutative BCK-algebras is not 1-based [10]. Recently, Padmanabhan and Rudeanu showed that the variety is 2-based, and gave the following explicit basis [9, Lemma 6].

$$(x \to x) \to y = y$$

$$(x \to (y \to z)) \to ((u \to v) \to v) = (y \to (x \to z)) \to ((v \to u) \to u)\,.$$

Here we offer the following particularly elegant improvement.

Theorem 1.2. *The following identities form a basis for the variety of commutative BCK-algebras:*

$$(x \to x) \to y = y \tag{C1}$$

$$(x \to y) \to (z \to y) = (y \to x) \to (z \to x)\,. \tag{C2}$$

Commutative BCK-algebras have a natural upper semilattice structure defined by $x \vee y = (x \to y) \to y$. The constant 1 is the top element of this semilattice. A commutative BCK-algebra is *bounded* if there is also a bottom element 0. D. Mundici showed that MV-algebras and bounded, commutative BCK-algebras are term equivalent [8]

A commutative BCK-algebra $(A, \to, 1)$ (or $(A, \to)$) is said to be an *ŁBCK-algebra* ("Ł" for Łukasiewicz) if it is a $\to$-subreduct of a bounded, commutative BCK-algebra $(A, \to, 1, 0)$ (or $(A, \to, 0)$) [6]. The class of ŁBCK-algebras is a subvariety of the variety of commutative BCK-algebras axiomatized by (B1)–(B4) (or the equivalent forms after removing 1) and

$$(x \to y) \to (y \to x) = y \to x\,. \tag{B5}$$

By Theorem 1.2, a 3-base for ŁBCK-algebras is given by (C1), (C2) and (B5). However, there is also a nice 2-base.

Theorem 1.3. *The following identities form a basis for the variety of ŁBCK-algebras:*

$$(x \to x) \to y = y \tag{L1}$$

$$((x \to y) \to (z \to x)) \to (y \to x) = (x \to z) \to (y \to z)\,. \tag{L2}$$

In §2, §3 and §4, we prove Theorems 1.1, 1.2 and 1.3, respectively. Finally in §5, we give some open problems.

2. MV-algebras

In this section we prove Theorem 1.1. First we show that MV-algebras satisfy the identities (M1) and (M2).

Lemma 2.1. *Every MV-algebra satisfies (M1) and (M2).*

Proof. First, we observe that

$$x \oplus \neg x \overset{(A2)}{=} \neg x \oplus x = \neg 0\,. \tag{2.1}$$

Indeed,

$$\begin{aligned}
\neg x \oplus x &\overset{(A3)}{=} \neg(\underbrace{x \oplus 0}) \oplus x \\
&\overset{(A2)}{=} \neg(0 \oplus x) \oplus x \\
&\overset{(A4)}{=} \neg(\neg\neg 0 \oplus x) \oplus x \\
&\overset{(A6)}{=} \neg(\neg x \oplus \neg 0) \oplus \neg 0 \\
&\overset{(A5)}{=} \neg 0\,.
\end{aligned}$$

For (M1), we have

$$\begin{aligned}
\neg(x \oplus (\neg x \oplus y)) \oplus z &\overset{(A1)}{=} \neg((x \oplus \neg x) \oplus y) \oplus z \\
&\overset{(2.1)}{=} \neg(\underbrace{\neg 0 \oplus y}) \oplus z \\
&\overset{(A2)}{=} \neg(y \oplus \neg 0) \oplus z \\
&\overset{(A5)}{=} \neg\neg 0 \oplus z \\
&\overset{(A4)}{=} 0 \oplus z \\
&\overset{(A2)}{=} z \oplus 0 \\
&\overset{(A3)}{=} z\,.
\end{aligned}$$

For (M2), we compute

$$
\begin{aligned}
&\neg(\neg(x\oplus y)\oplus\neg(z\oplus u))\oplus\neg(z\oplus(u\oplus\neg(\underbrace{x\oplus(y\oplus(z\oplus u))})))\\
&\overset{(A1)}{=} \neg(\neg(x\oplus y)\oplus\neg(z\oplus u))\oplus\neg((z\oplus u)\oplus\neg(\underbrace{(x\oplus y)}\oplus(z\oplus u)))\\
&\overset{(A4)}{=} \neg(\neg(x\oplus y)\oplus\neg(z\oplus u))\oplus\neg(\underbrace{(z\oplus u)\oplus\neg(\neg\neg(x\oplus y)\oplus(z\oplus u))})\\
&\overset{(A2)}{=} \neg(\neg(x\oplus y)\oplus\neg(z\oplus u))\oplus\neg(\underbrace{\neg(\neg\neg(x\oplus y)\oplus(z\oplus u))\oplus(z\oplus u)})\\
&\overset{(A6)}{=} \neg(\underbrace{\neg(x\oplus y)\oplus\neg(z\oplus u)})\oplus\neg(\neg(\neg(z\oplus u)\oplus\neg(x\oplus y))\oplus\neg(x\oplus y))\\
&\overset{(A2)}{=} \neg(\neg(z\oplus u)v\neg(x\oplus y))\oplus\neg(\neg(\neg(z\oplus u)\oplus\neg(x\oplus y))\oplus\neg(x\oplus y))\\
&\overset{(A2)}{=} \neg(\underbrace{\neg(\neg(z\oplus u)\oplus\neg(x\oplus y))\oplus\neg(x\oplus y)})\oplus\neg(\neg(z\oplus u)\oplus\neg(x\oplus y))\\
&\overset{(A2)}{=} \neg(\neg(x\oplus y)\oplus\neg(\neg(z\oplus u)\oplus\neg(x\oplus y)))\oplus\neg(\neg(z\oplus u)\oplus\neg(x\oplus y))\\
&\overset{(A6)}{=} \neg(\underbrace{\neg\neg(\neg(z\oplus u)\oplus\neg(x\oplus y))}\oplus(x\oplus y))\oplus(x\oplus y)\\
&\overset{(A4)}{=} \neg(\underbrace{(\neg(z\oplus u)\oplus\neg(x\oplus y))\oplus(x\oplus y)})\oplus(x\oplus y)\\
&\overset{(A1)}{=} \neg(\neg(z\oplus u)\oplus(\underbrace{\neg(x\oplus y)\oplus(x\oplus y)}))\oplus(x\oplus y)\\
&\overset{(2.1)}{=} \neg(\underbrace{\neg(z\oplus u)\oplus\neg 0})\oplus(x\oplus y)\\
&\overset{(A5)}{=} \neg\neg 0\oplus(x\oplus y)\\
&\overset{(A4)}{=} 0\oplus(x\oplus y)\\
&\overset{(A2)}{=} (x\oplus y)\oplus 0\\
&\overset{(A3)}{=} x\oplus y\,.
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 2.2. *Let $(A,\oplus,\neg)$ be an algebra satisfying (M1) and (M2). Then $(A,\oplus,\neg)$ is an MV-algebra.*

Proof. By (M1), any expression of the form $\neg(x\oplus(\neg x\oplus y))$ is a left identity element for $\oplus$. We denote this expression by $e_{x,y}$ so that

$$e_{x,y}\oplus z = z\,. \tag{2.2}$$

Our first step is to give two simpler consequences of (M2) which we will use in the rest of the proof rather than (M2) itself. Now in (M2), set $x = e_{w,w}$ and use (2.2) three times to get

$$\neg(\neg y\oplus\neg(z\oplus u))\oplus\neg(z\oplus(u\oplus\neg(y\oplus(z\oplus u)))) = y\,. \tag{2.3}$$

Also, set $z = e_{w,w}$ in (M2), and use (2.2) three times to get

$$\neg(\neg(x \oplus y) \oplus \neg u) \oplus \neg(u \oplus \neg(x \oplus (y \oplus u))) = x \oplus y\,. \tag{2.4}$$

In the next step, we determine the constant 0. First, set $z = e_{w,w}$ in (2.3) and use (2.2) three times to get

$$\neg(\neg y \oplus \neg u) \oplus \neg(u \oplus \neg(y \oplus u)) = y\,. \tag{2.5}$$

Now set $y = e_{x,z}$ in (2.5) reversed to get

$$\begin{aligned} e_{x,z} &= \neg(\neg e_{x,z} \oplus \neg u) \oplus \neg(u \oplus \neg(\underbrace{e_{x,z} \oplus u}))\\ &\overset{(2.2)}{=} \neg(\underbrace{\neg e_{x,z} \oplus \neg u}) \oplus \neg(u \oplus \neg u)\\ &\overset{(2.2)}{=} \neg(e_{x,z} \oplus (\neg e_{x,z} \oplus \neg u)) \oplus \neg(u \oplus \neg u)\\ &= e_{e_{x,z},\neg u} \oplus \neg(u \oplus \neg u)\\ &\overset{(2.2)}{=} \neg(u \oplus \neg u)\,, \end{aligned}$$

which gives,

$$\neg(u \oplus \neg u) = e_{x,z}\,.$$

The left side of this last equation does not depend on x and z, and the right side does not depend on u, so both sides are constant. Thus we now define

$$0 = \neg(u \oplus \neg u) = e_{x,z}\,. \tag{2.6}$$

Now we turn to the axioms themselves, starting with (A3). By (M1) (or (2.2)), we have

$$0 \oplus x = x\,, \tag{2.7}$$

which is almost (A3). In (2.3), take $u = \neg z$ and apply (2.6) twice to get

$$\neg(\neg y \oplus 0) \oplus 0 = y\,. \tag{2.8}$$

Set $u = \neg y$ in (2.5) reversed to obtain

$$\begin{aligned} y &= \underbrace{\neg(\neg y \oplus \neg\neg y)} \oplus \neg(\neg y \oplus \underbrace{\neg(y \oplus \neg y)})\\ &\overset{(2.6)}{=} 0 \oplus \neg(\neg y \oplus \neg 0)\\ &\overset{(2.7)}{=} \neg(\neg y \oplus \neg 0)\,, \end{aligned}$$

which gives

$$\neg(\neg y \oplus 0) = y\,. \tag{2.9}$$

Using this in the left side of (2.8), we have $y \oplus 0 = y$, which is (A3).

Applying (A3) to (2.9), we obtain $\neg\neg y = y$, which is (A4).

Now

$$0 = e_{0,x} = \neg(0 \oplus (\neg 0 \oplus x)) \overset{(2.7)}{=} \neg(\neg 0 \oplus x)\,.$$

So applying $\neg$ to both sides of this and using (A4), we have

$$\neg 0 \oplus x = \neg 0\,, \tag{2.10}$$

which is almost (A5).

To prove (A5) itself, we compute

$$\begin{aligned} x \oplus \neg 0 &\overset{(2.6)}{=} x \oplus \neg\neg(\neg x \oplus \underbrace{\neg\neg x}) \\ &\overset{(A4)}{=} x \oplus \underbrace{\neg\neg(\neg x \oplus x)} \\ &\overset{(A4)}{=} x \oplus (\neg x \oplus x) \\ &\overset{(A4)}{=} \neg\neg(x \oplus (\neg x \oplus x)) \\ &= \neg e_{x,x} \\ &\overset{(2.6)}{=} \neg 0\,, \end{aligned}$$

thus establishing the claim.

The next and longest part of the proof is of commutativity (A2).

Set $u = 0$ in (2.3), apply (A3) twice and (2.7) once to obtain

$$\neg(\neg y \oplus \neg z) \oplus \neg(z \oplus \neg(y \oplus z)) = y\,.$$

Adding $\neg y \oplus \neg z$ on the left to both sides of this and reversing, we get

$$\begin{aligned} (\neg y \oplus \neg z) \oplus y &= (\neg y \oplus \neg z) \oplus [\neg(\neg y \oplus \neg z) \oplus \neg(z \oplus \neg(y \oplus z))] \\ &= \neg e_{\neg y \oplus \neg z, \neg(z \oplus \neg(y \oplus z))} \\ &\overset{(2.6)}{=} \neg 0\,. \end{aligned}$$

Setting $y = \neg x$ and $z = \neg y$, and applying (A4) twice, we obtain

$$(x \oplus y) \oplus \neg x = \neg 0\,. \tag{2.11}$$

In (2.5), set $y = x \oplus z$ and $u = \neg x$. Then $y \oplus u = \neg 0$ by (2.11), and so (2.5) reversed becomes

$$\begin{aligned} x \oplus z &= \neg(\neg(x \oplus z) \oplus \underbrace{\neg\neg x}) \oplus \neg(\neg x \oplus \underbrace{\neg\neg 0}) \\ &\overset{(A4)}{=} \neg(\neg(x \oplus z) \oplus x) \oplus \neg(\underbrace{\neg x \oplus 0}) \\ &\overset{(A3)}{=} \neg(\neg(x \oplus z) \oplus x) \oplus \underbrace{\neg\neg x} \\ &\overset{(A4)}{=} \neg(\neg(x \oplus z) \oplus x) \oplus x\,. \end{aligned}$$

Replacing z with y and reversing this gives

$$\neg(\neg(x \oplus y) \oplus x) \oplus x = x \oplus y\,. \tag{2.12}$$

In (2.3), set $y = 0$. Then in reverse, (2.3) becomes

$$\begin{aligned}
0 &= \neg(\underbrace{\neg 0 \oplus \neg(z \oplus u)}) \oplus \neg(z \oplus (u \oplus \neg(0 \oplus (z \oplus u)))) \\
&\overset{(2.10)}{=} \neg\neg 0 \oplus \neg(z \oplus (u \oplus \neg(\underbrace{0 \oplus (z \oplus u)}))) \\
&\overset{(2.7)}{=} \underbrace{\neg\neg 0} \oplus \neg(z \oplus (u \oplus \neg(z \oplus u))) \\
&\overset{(A4)}{=} 0 \oplus \neg(z \oplus (u \oplus \neg(z \oplus u))) \\
&\overset{(2.7)}{=} \neg(z \oplus (u \oplus \neg(z \oplus u)))
\end{aligned}$$

Replacing z with x, u with y and reversing this, we have

$$\neg(x \oplus (y \oplus \neg(x \oplus y))) = 0\,. \tag{2.13}$$

In (2.3), let $y = \neg x$ and $z = x$. Then $\neg y = x$ by (A4), and so (2.3) reversed becomes

$$\begin{aligned}
\neg x &= \neg(x \oplus \neg(x \oplus u)) \oplus \neg(x \oplus (u \oplus \neg(\underbrace{\neg x \oplus (x \oplus z)}))) \\
&\overset{(2.6)}{=} \neg(x \oplus \neg(x \oplus u)) \oplus \neg(x \oplus (u \oplus \underbrace{\neg\neg 0})) \\
&\overset{(A4)}{=} \neg(x \oplus \neg(x \oplus u)) \oplus \neg(x \oplus (\underbrace{u \oplus 0})) \\
&\overset{(A3)}{=} \neg(x \oplus \neg(x \oplus u)) \oplus \neg(x \oplus u)\,.
\end{aligned}$$

Reversing this and replacing u with y, we have

$$\neg(x \oplus \neg(x \oplus y)) \oplus \neg(x \oplus y) = \neg x\,. \tag{2.14}$$

In (2.5), set $u = \neg(y \oplus z)$. Then $\neg(\neg y \oplus \neg u) = \neg(\neg y \oplus (y \oplus z)) = e_{\neg y, y} = 0$ by (A4) and (2.6), and so (2.5) reversed becomes

$$\begin{aligned}
y &= 0 \oplus \neg(\neg(y \oplus z) \oplus \neg(y \oplus \neg(y \oplus z))) \\
&\overset{(2.7)}{=} \neg(\neg(y \oplus z) \oplus \neg(y \oplus \neg(y \oplus z)))\,.
\end{aligned}$$

Apply (A4) to both sides of this, replace y with x, z with y, and reverse to obtain

$$\neg(x \oplus y) \oplus \neg(x \oplus \neg(x \oplus y)) = \neg x\,. \tag{2.15}$$

In (2.4), replace x with $\neg x$, y with $\neg y$ and set $u = y \oplus \neg(x \oplus y)$. Then $\neg(\neg(\neg x \oplus \neg y) \oplus \neg u) = \neg x$ by (2.5), and so (2.4) reversed becomes

$$\begin{aligned}
\neg x \oplus \neg y &= \neg x \oplus \neg((y \oplus \neg(x \oplus y)) \oplus \neg(\neg x \oplus \neg(\underbrace{\neg y \oplus (y \oplus \neg(x \oplus y))}))) \\
&\overset{(2.6)}{=} \neg x \oplus \neg((y \oplus \neg(x \oplus y)) \oplus \neg(\underbrace{\neg x \oplus \neg 0})) \\
&\overset{(A5)}{=} \neg x \oplus \neg((y \oplus \neg(x \oplus y)) \oplus \neg\neg 0) \\
&\overset{(A4)}{=} \neg x \oplus \neg(\underbrace{(y \oplus \neg(x \oplus y)) \oplus 0}) \\
&\overset{(A3)}{=} \neg x \oplus \neg(y \oplus \neg(x \oplus y))
\end{aligned}$$

Replacing x with $\neg x$ once again and using (A4), our last calculation yields

$$x \oplus \neg(y \oplus \neg(\neg x \oplus y)) = x \oplus \neg y\,. \tag{2.16}$$

In (2.5), set $y = x$ and $u = \neg(y \oplus \neg(\neg x \oplus y))$. Then $\neg u = y \oplus \neg(\neg x \oplus y)$ by (A4), and so (2.5) reversed becomes

$$\begin{aligned}
x &= \neg[\neg y \oplus (y \oplus \neg(\neg x \oplus y))] \oplus \neg\{\neg(y \oplus \neg(\neg x \oplus y)) \oplus \neg[\underbrace{x \oplus \neg(y \oplus \neg(\neg x \oplus y))}]\} \\
&\overset{(2.16)}{=} \underbrace{\neg[\neg y \oplus (y \oplus \neg(\neg x \oplus y))]} \oplus \neg\{\neg(y \oplus \neg(\neg x \oplus y)) \oplus \neg[x \oplus \neg y]\} \\
&\overset{(2.13)}{=} 0 \oplus \neg\{\neg(y \oplus \neg(\neg x \oplus y)) \oplus \neg[x \oplus \neg y]\} \\
&\overset{(2.7)}{=} \neg\{\neg(y \oplus \neg(\neg x \oplus y)) \oplus \neg[x \oplus \neg y]\}
\end{aligned}$$

Now apply (A4) to both sides of this last calculation and exchange the roles of x and y to obtain

$$\neg(x \oplus \neg(\neg y \oplus x)) \oplus \neg(y \oplus \neg x) = \neg y\,. \tag{2.17}$$

Next, we compute

$$\begin{aligned}
\neg 0 &\overset{(2.6)}{=} \neg e_{x\oplus(\neg y\oplus x),\neg(y\oplus\neg x)} \\
&\overset{(A4)}{=} (x \oplus (\neg y \oplus x)) \oplus [\underbrace{\neg(x \oplus (\neg y \oplus x)) \oplus \neg(y \oplus \neg x)}] \\
&\overset{(2.17)}{=} (x \oplus (\neg y \oplus x)) \oplus \neg y\,,
\end{aligned}$$

and so we have shown

$$(x \oplus \neg(\neg y \oplus x)) \oplus \neg y = \neg 0\,. \tag{2.18}$$

In (2.16), take $x = u \oplus \neg(u \oplus v)$ and $y = \neg(u \oplus v)$. Then $\neg x \oplus y = \neg u$ by (2.14), and so the left side of (2.16) becomes

$$(u \oplus \neg(u \oplus v)) \oplus \neg(\neg(u \oplus v) \oplus \underbrace{\neg\neg u}) \overset{(A4)}{=} (u \oplus \neg(u \oplus v)) \oplus \neg(\neg(u \oplus v) \oplus u)\,.$$

The right side of (2.16) becomes

$$(u \oplus \neg(u \oplus v)) \oplus \underbrace{\neg\neg(u \oplus v)} \overset{(A4)}{=} (u \oplus \neg(u \oplus v)) \oplus (u \oplus v)\,.$$

Replacing u with x and v with y, we now have

$$(x \oplus \neg(x \oplus y)) \oplus \neg(\neg(x \oplus y) \oplus x) = (x \oplus \neg(x \oplus y)) \oplus (x \oplus y)\,. \tag{2.19}$$

In (2.18), take $y = \neg(x \oplus z) \oplus x$. Then $\neg y \oplus x = x \oplus z$ by (2.12), and so (2.18) reversed becomes

$$\begin{aligned}
\neg 0 &= (x \oplus \neg(x \oplus z)) \oplus \underbrace{\neg(\neg(x \oplus z) \oplus x)} \\
&\overset{(2.19)}{=} (x \oplus \neg(x \oplus z)) \oplus (x \oplus z)\,.
\end{aligned}$$

Replacing z with y and reversing, we have

$$(x \oplus \neg(x \oplus y)) \oplus (x \oplus y) = \neg 0\,. \tag{2.20}$$

In (2.17), take $x = u \oplus v$ and $y = \neg(u \oplus \neg(u \oplus v))$. Then $y \oplus \neg x = \neg u$ by (2.14) and $\neg y = u \oplus \neg(u \oplus v))$ by (A4), and so (2.17) reversed becomes

$$\begin{aligned} u \oplus \neg(u \oplus v) &= \neg[(u \oplus v) \oplus (\underbrace{\neg\neg(u \oplus \neg(u \oplus v))} \oplus (u \oplus v))] \oplus \underbrace{\neg\neg u} \\ &\overset{(A4)}{=} \neg[(u \oplus v) \oplus (\underbrace{(u \oplus \neg(u \oplus v)) \oplus (u \oplus v)})] \oplus u \\ &\overset{(2.20)}{=} \neg[(u \oplus v) \oplus \underbrace{\neg\neg 0}] \oplus u \\ &\overset{(A4)}{=} \neg[\underbrace{(u \oplus v) \oplus 0}] \oplus u \\ &\overset{(A3)}{=} \neg(u \oplus v) \oplus u\,. \end{aligned}$$

Replace u with x and v with y, and reverse to get

$$\neg(x \oplus y) \oplus x = x \oplus \neg(x \oplus y)\,. \tag{2.21}$$

In (2.21), take $x = \neg(u \oplus (\neg v \oplus u))$ and $y = \neg(v \oplus \neg u)$. Then $\neg(x \oplus y) = v$ by (2.17). Thus (2.21) reversed becomes

$$\neg(u \oplus \neg(\neg v \oplus u)) \oplus v = v \oplus \underbrace{\neg(u \oplus \neg(\neg v \oplus u))} \overset{(2.16)}{=} v \oplus \neg u\,.$$

Replacing u with x and v with y, we now have

$$\neg(x \oplus \neg(\neg y \oplus x)) \oplus y = y \oplus \neg x\,. \tag{2.22}$$

Next, in (2.15), set $x = \neg(v \oplus \neg(\neg u \oplus v))$ and $y = \neg(u \oplus \neg v)$. Then by (2.17), $\neg(x \oplus y) = u$, and so the right side of (2.15) becomes

$$\neg\neg(v \oplus \neg(\neg u \oplus v)) \overset{(A4)}{=} v \oplus \neg(\neg u \oplus v)\,.$$

The left side of (2.15) is

$$u \oplus \neg(\neg(v \oplus \neg(\neg u \oplus v)) \oplus u) \overset{(2.22)}{=} u \oplus \neg(u \oplus \neg v)\,.$$

So replacing u with x and v with y, we have obtained

$$x \oplus \neg(\neg y \oplus x) = y \oplus \neg(y \oplus \neg x)\,. \tag{2.23}$$

Next, we compute

$$\begin{aligned} x \oplus (y \oplus \neg(y \oplus \underbrace{\ x\ })) &\overset{(A4)}{=} x \oplus (\underbrace{y \oplus \neg(y \oplus \neg\neg x)}) \\ &\overset{(2.23)}{=} x \oplus (\neg x \oplus \neg(\neg y \oplus \neg x)) \\ &= \neg e_{x,\neg(\neg y \oplus \neg x)} \\ &\overset{(2.6)}{=} \neg 0\,, \end{aligned}$$

which gives us

$$x \oplus (y \oplus \neg(y \oplus x)) = \neg 0\,. \tag{2.24}$$

Now in (2.4), take $u = \neg(y \oplus x)$. We have

$$\begin{aligned}
x \oplus y &= \neg(\neg(x \oplus y) \oplus \neg\neg(y \oplus x)) \oplus \neg(\neg(y \oplus x) \oplus \neg(\underbrace{x \oplus (y \oplus \neg(y \oplus x))}))\\
&\overset{(2.24)}{=} \neg(\neg(x \oplus y) \oplus \underbrace{\neg\neg(y \oplus x)}) \oplus \neg(\neg(y \oplus x) \oplus \underbrace{\neg\neg 0})\\
&\overset{(A4)}{=} \neg(\neg(x \oplus y) \oplus (y \oplus x)) \oplus \neg(\underbrace{\neg(y \oplus x) \oplus 0})\\
&\overset{(A3)}{=} \neg(\neg(x \oplus y) \oplus (y \oplus x)) \oplus \underbrace{\neg\neg(y \oplus x)}\\
&\overset{(A4)}{=} \neg(\neg(x \oplus y) \oplus (y \oplus x)) \oplus (y \oplus x)\,,
\end{aligned}$$

and hence we obtain

$$\neg(\neg(x \oplus y) \oplus (y \oplus x)) \oplus (y \oplus x) = x \oplus y\,. \tag{2.25}$$

Now in (2.3), take $y = \neg(\neg(u \oplus z) \oplus (z \oplus u))$. Then $\neg(\neg(u \oplus z) \oplus (z \oplus u)) =$

$$\begin{aligned}
&= \neg[(\neg(u \oplus z) \oplus (z \oplus u)) \oplus \neg(z \oplus u)] \oplus \neg(z \oplus (u \oplus \neg(\underbrace{\neg(\neg(u \oplus z) \oplus (z \oplus u)) \oplus (z \oplus u)})))\\
&\overset{(2.25)}{=} \neg[(\neg(u \oplus z) \oplus (z \oplus u)) \oplus \neg(z \oplus u)] \oplus \neg(\underbrace{z \oplus (u \oplus \neg(u \oplus z))})\\
&\overset{(2.24)}{=} \neg[(\neg(u \oplus z) \oplus (z \oplus u)) \oplus \neg(z \oplus u)] \oplus \underbrace{\neg\neg 0}\\
&\overset{(A4)}{=} \neg[(\neg(u \oplus z) \oplus (z \oplus u)) \oplus \neg(z \oplus u)] \oplus 0\\
&\overset{(A3)}{=} \neg[(\neg(u \oplus z) \oplus (z \oplus u)) \oplus \neg(z \oplus u)]\,.
\end{aligned}$$

Now apply (A4) to each side, and replace u with x and z with y to obtain

$$(\neg(x \oplus y) \oplus (y \oplus x)) \oplus \neg(y \oplus x) = \neg(x \oplus y) \oplus (y \oplus x)\,. \tag{2.26}$$

Next, in (2.23), set $x = v \oplus u$ and $y = \neg(u \oplus v) \oplus (v \oplus u)$. The left side becomes

$$(v \oplus u) \oplus \neg[\neg(\neg(u \oplus v) \oplus (v \oplus u)) \oplus (v \oplus u)] \overset{(2.25)}{=} (v \oplus u) \oplus \neg(u \oplus v)\,.$$

The right side of (2.23) becomes

$$\begin{aligned}
&[\neg(u \oplus v) \oplus (v \oplus u)] \oplus \neg(\underbrace{[\neg(u \oplus v) \oplus (v \oplus u)] \oplus \neg(v \oplus u)})\\
&\overset{(2.26)}{=} [\neg(u \oplus v) \oplus (v \oplus u)] \oplus \neg[\neg(u \oplus v) \oplus (v \oplus u)]\\
&\overset{(A4)}{=} \neg\neg\{[\neg(u \oplus v) \oplus (v \oplus u)] \oplus \neg[\neg(u \oplus v) \oplus (v \oplus u)]\}\\
&\overset{(2.6)}{=} \neg 0\,.
\end{aligned}$$

Thus, we have

$$(v \oplus u) \oplus \neg(u \oplus v) = \neg 0\,. \tag{2.27}$$

Next, in (2.24), set $x = \neg(u \oplus v)$ and $z = v \oplus u$, and reverse to obtain

$$\begin{aligned}
\neg 0 &= \neg(u \oplus v) \oplus [(v \oplus u) \oplus \neg(\underbrace{(v \oplus u) \oplus \neg(u \oplus v)})] \\
&\overset{(2.27)}{=} \neg(u \oplus v) \oplus [(v \oplus u) \oplus \underbrace{\neg\neg 0}] \\
&\overset{(A4)}{=} \neg(u \oplus v) \oplus [\underbrace{(v \oplus u) \oplus 0}] \\
&\overset{(A3)}{=} \neg(u \oplus v) \oplus (v \oplus u)\,,
\end{aligned}$$

that is,

$$\neg(u \oplus v) \oplus (v \oplus u) = \neg 0\,. \tag{2.28}$$

Now we apply (2.28) to (2.25) to get

$$x \oplus y = \underbrace{\neg\neg 0} \oplus (y \oplus x) \overset{(A4)}{=} 0 \oplus (y \oplus x) \overset{(2.7)}{=} y \oplus x\,.$$

We have therefore established the commutativity of $\oplus$, that is, (A2).

Applying (A2) to the left side of (2.23) once and the right side twice, we obtain (A6). All that remains is to establish associativity (A1).

Adding $u \oplus (\neg(x \oplus (y \oplus u)))$ to the left on both sides of (2.4) and reversing, we get

$$\begin{aligned}
&(u \oplus (\neg(x \oplus (y \oplus u)))) \oplus (x \oplus y) \\
&= (u \oplus (\neg(x \oplus (y \oplus u)))) \oplus [\underbrace{\neg(\neg(x \oplus y) \oplus \neg u) \oplus \neg(u \oplus (\neg(x \oplus (y \oplus u))))}] \\
&\overset{(A2)}{=} (u \oplus (\neg(x \oplus (y \oplus u)))) \oplus [\neg(u \oplus (\neg(x \oplus (y \oplus u)))) \oplus \neg(\neg(x \oplus y) \oplus \neg u)] \\
&= \neg e_{u \oplus (\neg(x \oplus (y \oplus u))), \neg(\neg(x \oplus y) \oplus \neg u)} \\
&\overset{(2.6)}{=} \neg 0\,.
\end{aligned}$$

Replacing u with z, this gives

$$(z \oplus (\neg(x \oplus (y \oplus z)))) \oplus (x \oplus y) = \neg 0\,.$$

Rearranging this using (A2), we have

$$(x \oplus y) \oplus (z \oplus (\neg(x \oplus (y \oplus z)))) = \neg 0\,. \tag{2.29}$$

In (2.29), take $x = v \oplus u$ and $z = \neg(v \oplus (u \oplus y))$. Then $x \oplus (y \oplus z) = \neg 0$ by (2.29), and so with the new variables, (2.29) reversed becomes

$$\begin{aligned}
\neg 0 &= ((v \oplus u) \oplus y) \oplus (\neg(v \oplus (u \oplus y)) \oplus \underbrace{\neg\neg 0}) \\
&\overset{(A4)}{=} ((v \oplus u) \oplus y) \oplus (\underbrace{\neg(v \oplus (u \oplus y)) \oplus 0}) \\
&\overset{(A3)}{=} ((v \oplus u) \oplus y) \oplus \neg(v \oplus (u \oplus y))\,.
\end{aligned}$$

Replacing v with x, u with y and y with z, this gives

$$((x \oplus y) \oplus z) \oplus \neg(x \oplus (y \oplus z)) = \neg 0\,. \tag{2.30}$$

Set $x = (u \oplus v) \oplus w$ and $y = u \oplus (v \oplus w)$. Then

$$x \oplus \neg y = ((u \oplus v) \oplus w) \oplus \neg(u \oplus (v \oplus w)) \overset{(2.30)}{=} \neg 0\,,$$

and also

$$\begin{aligned}
y \oplus \neg x &= (\underbrace{u \oplus (v \oplus w)}) \oplus \neg(\underbrace{(u \oplus v) \oplus w}) \\
&\overset{(A2)}{=} ((\underbrace{v \oplus w}) \oplus u) \oplus \neg(w \oplus (u \oplus v)) \\
&\overset{(A2)}{=} ((v \oplus w) \oplus u) \oplus \neg(w \oplus (u \oplus v)) \\
&\overset{(2.30)}{=} \neg 0 \,.
\end{aligned}$$

Now using (A2), rewrite (A6) as $x \oplus \neg(x \oplus \neg y) = y \oplus \neg(y \oplus \neg x)$. With x and y as above, this is

$$[(u \oplus v) \oplus w] \oplus \neg\neg 0 = (u \oplus (v \oplus w)) \oplus \neg\neg 0 \,.$$

Applying (A4) to both sides followed by (A3), we have $(u \oplus v) \oplus w = u \oplus (v \oplus w)$, that is, we have associativity (A1). □

Putting Lemmas 2.1 and 2.2 together, we almost have Theorem 1.1. All that remains is to check the independence of (M1) and (M2). We just give the models, leaving the detailed verifications to the reader.

On a 2-element set $\{0, 1\}$, define $x \oplus 0 = 0 \oplus x = 0$ for all x, $1 \oplus 1 = 1$ and $\neg x = 1$ for all x. This model satisfies (M1), but not (M2).

On a 2-element set $\{0, 1\}$, define $x \oplus y = 1$ and $\neg x = 0$ for all x, y. This model satisfies (M2), but not (M1).

Remark 2.3. Note that (2.3) and (2.4) were the only direct consequences of (M2) used in the proof. Thus we have also shown that (M1), (2.3) and (2.4) is a 3-base for MV-algebras.

3. Commutative BCK-Algebras

In this section we prove Theorem 1.2. We start with the easy direction.

Lemma 3.1. *Every commutative BCK-algebra satisfies* (C1) *and* (C2).

Proof. (C1) follows immediately from (B3) and (B4). For (C2), we have

$$\begin{aligned}
(x \to y) \to (z \to y) &\overset{(B2)}{=} z \to ((x \to y) \to y) \\
&\overset{(B1)}{=} z \to ((y \to x) \to x) \\
&\overset{(B2)}{=} (y \to x) \to (z \to x) \,,
\end{aligned}$$

as claimed. □

Lemma 3.2. *Let* $(A, \to)$ *be an algebra satisfying* (C1) *and* (C2). *Then* $(A, \to)$ *is a commutative BCK-algebra.*

Proof. First, we show that $x \to x$ is a constant, that is, $x \to x = y \to y$. Indeed,

$$
\begin{aligned}
x \to x &\overset{(C1)}{=} (y \to y) \to (x \to x) \\
&\overset{(C1)}{=} \underbrace{(x \to x)} \to [(y \to y) \to (x \to x)] \\
&\overset{(C1)}{=} [(y \to y) \to (x \to x)] \to [(y \to y) \to (x \to x)] \\
&\overset{(C2)}{=} [\underbrace{(x \to x) \to (y \to y)}] \to [\underbrace{(y \to y) \to (y \to y)}] \\
&\overset{(C1)}{=} (y \to y) \to (y \to y) \\
&\overset{(C1)}{=} y \to y\,.
\end{aligned}
$$

We now define $1 = x \to x$ and note that this definition and (C1) give (B3) and (B4). (B1) then follows from taking $z = 1$ in (C2) and using (B4) on both sides.

To prove (B2), we need the following identities:

$$x \to 1 = 1 \tag{3.1}$$

$$x \to (y \to x) = 1\,. \tag{3.2}$$

For (3.1), we compute

$$
\begin{aligned}
x \to 1 &\overset{(B4)}{=} (1 \to x) \to 1 \\
&\overset{(B3)}{=} (1 \to x) \to (x \to x) \\
&\overset{(C2)}{=} (x \to 1) \to (x \to 1) \overset{(B3)}{=} 1\,.
\end{aligned}
$$

For (3.2), we compute

$$
\begin{aligned}
x \to (y \to x) &\overset{(B4)}{=} (1 \to x) \to (y \to x) \\
&\overset{(C2)}{=} (x \to 1) \to (y \to 1) \\
&\overset{(3.1)}{=} 1 \to (y \to 1) \\
&\overset{(3.1)}{=} 1 \to 1 \\
&\overset{(B4)}{=} 1\,.
\end{aligned}
$$

Next we show

$$(x \to (y \to z)) \to (y \to (x \to z)) = 1\,. \tag{3.3}$$

Indeed, we have

$$
\begin{aligned}
(\underbrace{x \to (y \to z)}) \to (y \to (x \to z)) &\overset{(B4)}{=} [\underbrace{1} \to (x \to (y \to z))] \to (y \to (x \to z)) \\
&\overset{(3.2)}{=} [\underbrace{(z \to (y \to z)) \to (x \to (y \to z))}] \to (y \to (x \to z)) \\
&\overset{(C2)}{=} [((y \to z) \to z) \to (x \to z)] \to (y \to (x \to z)) \\
&\overset{(C2)}{=} [(x \to z) \to ((y \to z) \to z)] \to [y \to (\underbrace{(y \to z) \to z})] \\
&\overset{(B1)}{=} [(x \to z) \to ((y \to z) \to z)] \to [y \to ((z \to y) \to y)] \\
&\overset{(3.2)}{=} [(x \to z) \to ((y \to z) \to z)] \to 1 \\
&\overset{(3.1)}{=} 1\,.
\end{aligned}
$$

Finally, we prove (B2) as follows:

$$
\begin{aligned}
x \to (y \to z) &\overset{(B4)}{=} \underbrace{1} \to (x \to (y \to z)) \\
&\overset{(3.3)}{=} [(y \to (x \to z)) \to (x \to (y \to z))] \to (x \to (y \to z)) \\
&\overset{(B1)}{=} [\underbrace{(x \to (y \to z)) \to (y \to (x \to z))}] \to (y \to (x \to z)) \\
&\overset{(3.3)}{=} 1 \to (y \to (x \to z)) \\
&\overset{(B4)}{=} y \to (x \to z)\,.
\end{aligned}
$$

This completes the proof of the lemma. □

From Lemmas 3.1 and 3.2, we almost have Theorem 1.2, modulo checking the independence of (C1) and (C2). As before, we just give the models, leaving the details to the reader.

On a 2-element set $\{0, 1\}$, define $x \to 0 = 0$ and $x \to 1 = 1$ for all x. This model satisfies (C1), but not (C2).

On a 2-element set $\{0, 1\}$, define $x \to y = 1$ for all x, y. This model satisfies (C2), but not (C1).

4. ŁBCK-algebras

In this section we prove Theorem 1.3. As usual, we start with the easy direction.

Lemma 4.1. *Every ŁBCK-algebra satisfies* (L1) *and* (L2).

Proof. We will use not only (B1)–(B4), but also identities derived in §3. First, (L1) is just (C1), so Lemma 3.1 applies.

To obtain (L2) will require more work. First, we show

$$(x \to y) \to [((z \to (y \to x)) \to x) \to x] = 1\,. \tag{4.1}$$

Indeed, we have

$$
\begin{aligned}
&(x \to y) \to [((z \to (y \to x)) \to x) \to y] \\
&\overset{(C2)}{=} (y \to x) \to [((z \to (y \to x)) \to x) \to x] \\
&\overset{(B1)}{=} (y \to x) \to [(x \to (z \to (y \to x))) \to (z \to (y \to x))] \\
&\overset{(B2)}{=} (x \to (z \to (y \to x))) \to [(y \to x) \to (z \to (y \to x))] \\
&\overset{(3.2)}{=} (x \to (z \to (y \to x))) \to 1 \\
&\overset{(3.1)}{=} 1\,.
\end{aligned}
$$

Next, we show

$$
(((x \to y) \to z) \to ((y \to x) \to z)) \to z = (x \to y) \to z\,. \tag{4.2}
$$

Set $a = ((x \to y) \to z) \to ((y \to x) \to z)$ and $b = a \to z$. We compute

$$
\begin{aligned}
b &\overset{(C2)}{=} ((z \to (x \to y)) \to (\underbrace{(y \to x) \to (x \to y)})) \to z \\
&\overset{(B5)}{=} (\underbrace{(z \to (x \to y)) \to (x \to y)}) \to z \\
&\overset{(B1)}{=} (((x \to y) \to z) \to z) \to z \\
&\overset{(B1)}{=} (\underbrace{z \to ((x \to y) \to z)}) \to ((x \to y) \to z) \\
&\overset{(3.2)}{=} 1 \to ((x \to y) \to z) \\
&\overset{(B4)}{=} (x \to y) \to z\,.
\end{aligned}
$$

Now

$$
\begin{aligned}
(z \to (y \to x)) \to (b \to (y \to x)) &\overset{(C2)}{=} ((y \to x) \to z) \to (b \to z) \\
&= ((y \to x) \to z) \to (\underbrace{(a \to z) \to z}) \\
&\overset{(B1)}{=} ((y \to x) \to z) \to ((z \to a) \to a) \\
&\overset{(B2)}{=} (z \to a) \to (\underbrace{((y \to x) \to z) \to a}) \\
&\overset{(3.2)}{=} (z \to a) \to 1 \\
&\overset{(3.1)}{=} 1\,,
\end{aligned}
$$

that is,

$$
(z \to (y \to x)) \to (b \to (y \to x)) = 1\,. \tag{4.3}
$$

Also,

$$
\begin{aligned}
(b \to (y \to x)) \to (z \to (y \to x)) &\overset{(C2)}{=} ((y \to x) \to b) \to (\underbrace{z \to b}) \\
&\overset{(3.2)}{=} ((y \to x) \to b) \to 1 \\
&\overset{(3.1)}{=} 1\,,
\end{aligned}
$$

that is,

$$
(b \to (y \to x)) \to (z \to (y \to x)) = 1\,. \tag{4.4}
$$

Putting this together, we compute

$$
\begin{aligned}
((x \to y) \to z) \to (y \to x) &\overset{(4.2)}{=} b \to (y \to x) \\
&\overset{(B4)}{=} 1 \to (b \to (y \to x)) \\
&\overset{(4.3)}{=} [(z \to (y \to x)) \to (b \to (y \to x))] \to (b \to (y \to x)) \\
&\overset{(B1)}{=} [(b \to (y \to x)) \to (z \to (y \to x))] \to (z \to (y \to x)) \\
&\overset{(4.4)}{=} 1 \to (z \to (y \to x)) \\
&\overset{(B4)}{=} z \to (y \to x)\,,
\end{aligned}
$$

giving us

$$
((x \to y) \to z) \to (y \to x) = z \to (y \to x)\,. \tag{4.5}
$$

Finally, in (4.5), replace z with $z \to x$ to get

$$
((x \to y) \to (z \to x)) \to (y \to x) = (z \to x) \to (y \to x) \overset{(C2)}{=} (x \to z) \to (y \to z)\,.
$$

This establishes (L2). □

Lemma 4.2. *Let* $(A, \to)$ *be an algebra satisfying* (L1) *and* (L2)*. Then* $(A, \to)$ *is an LBCK-algebra.*

Proof. First, we establish

$$
[x \to (x \to x)] \to y = y \tag{4.6}
$$

by computing

$$
\begin{aligned}
[x \to (x \to x)] \to y &\overset{(L1)}{=} [((x \to x) \to x) \to (x \to x)] \to y \\
&\overset{(L1)}{=} [((x \to x) \to ((x \to x) \to x)) \to (x \to x)] \to y \\
&\overset{(L2)}{=} [(x \to (x \to x)) \to (x \to (x \to x))] \to y \\
&\overset{(L1)}{=} y\,.
\end{aligned}
$$

Next we verify (B1), we compute

$$\begin{aligned}
(x \to y) \to y &\overset{(L1)}{=} (x \to y) \to ((x \to x) \to y) \\
&\overset{(L2)}{=} (\underbrace{(x \to (x \to x)) \to (y \to x)}) \to ((x \to x) \to x) \\
&\overset{(4.6)}{=} (y \to x) \to (\underbrace{(x \to x) \to x}) \\
&\overset{(L1)}{=} (y \to x) \to x \,,
\end{aligned}$$

establishing the claim.

Our next goal is to show that the expression $x \to x$ is a constant. First, we have

$$\begin{aligned}
(x \to y) \to (x \to y) &\overset{(L1)}{=} [(y \to y) \to (x \to y)] \to (x \to y)] \\
&\overset{(B1)}{=} [(x \to y) \to (y \to y)] \to (y \to y) \\
&\overset{(L1)}{=} [((y \to y) \to (x \to y)) \to (y \to y)] \to (y \to y) \\
&\overset{(L2)}{=} [(y \to x) \to (y \to x)] \to (y \to y) \\
&\overset{(L1)}{=} y \to y \,,
\end{aligned}$$

that is,

$$(x \to y) \to (x \to y) = y \to y \,. \tag{4.7}$$

Next,

$$\begin{aligned}
x \to x &\overset{(4.7)}{=} (y \to x) \to (y \to x) \\
&\overset{(4.7)}{=} [\underbrace{(z \to x) \to (y \to x)}] \to [(z \to x) \to (y \to x)] \\
&\overset{(L2)}{=} [((z \to y) \to (x \to z)) \to (y \to z)] \to [\underbrace{(z \to x) \to (y \to x)}] \\
&\overset{(L2)}{=} [((z \to y) \to (x \to z)) \to (y \to z)] \to [((z \to y) \to (x \to z)) \to (y \to z)] \\
&\overset{(4.7)}{=} (y \to z) \to (y \to z) \\
&\overset{(4.7)}{=} z \to z \,,
\end{aligned}$$

which shows that $x \to x$ is a constant. We thus set

$$x \to x = 1 \,.$$

This is (B3), and then (L1) gives (B4).

Next,

$$1 \overset{(B3)}{=} (x \to 1) \to (x \to 1) \overset{(B3)}{=} (x \to (x \to x)) \to (x \to 1) \overset{(4.6)}{=} x \to 1 \,,$$

which establishes (3.1).

Next we prove (3.2) as follows:

$$\begin{aligned} x \to (y \to x) &\overset{(B4)}{=} (1 \to x) \to (y \to x) \\ &\overset{(L2)}{=} ((\underbrace{1 \to y}) \to (x \to 1)) \to (y \to 1) \\ &\overset{(B4)}{=} (y \to (\underbrace{x \to 1})) \to (y \to 1) \\ &\overset{(3.1)}{=} (y \to 1) \to (y \to 1) \\ &\overset{(B3)}{=} 1\,. \end{aligned}$$

Now we compute

$$\begin{aligned} (\underbrace{(x \to y) \to y}) \to x &\overset{(B1)}{=} ((y \to x) \to x) \to x \\ &\overset{(B1)}{=} (\underbrace{x \to (y \to x)}) \to (y \to x) \\ &\overset{(3.2)}{=} 1 \to (y \to x) \\ &\overset{(B4)}{=} y \to x\,, \end{aligned}$$

which shows

$$((x \to y) \to y) \to x = y \to x\,. \tag{4.8}$$

Next we prove (B5):

$$\begin{aligned} (x \to y) \to (y \to x) &\overset{(4.8)}{=} [\underbrace{((x \to y) \to (y \to x)) \to (y \to x)}] \to (x \to y) \\ &\overset{(L2)}{=} [(x \to y) \to (\underbrace{y \to y})] \to (x \to y) \\ &\overset{(B3)}{=} [\underbrace{(x \to y) \to 1}] \to (x \to y) \\ &\overset{(3.1)}{=} 1 \to (x \to y) \\ &\overset{(B4)}{=} x \to y\,. \end{aligned}$$

Now we compute

$$\begin{aligned} (\underbrace{(x \to y) \to y}) \to (z \to x) &\overset{(B1)}{=} ((y \to x) \to x) \to (z \to x) \\ &\overset{(L2)}{=} [((x \to y) \to z) \to (\underbrace{x \to (y \to x)})] \to [z \to (y \to x)] \\ &\overset{(3.2)}{=} [\underbrace{((x \to y) \to z) \to 1}] \to [z \to (y \to x)] \\ &\overset{(3.1)}{=} 1 \to [z \to (y \to x)] \\ &\overset{(B4)}{=} z \to (y \to x)\,, \end{aligned}$$

which shows

$$((x \to y) \to y) \to (z \to x) = z \to (y \to x)\,. \tag{4.9}$$

Finally, we verify (B2) as follows:

$$
\begin{aligned}
x \to (y \to z) &\overset{(4.9)}{=} ((z \to y) \to y) \to (x \to z) \\
&\overset{(4.9)}{=} ((z \to x) \to x) \to [\underbrace{((z \to y) \to y) \to z}] \\
&\overset{(4.8)}{=} ((z \to y) \to y) \to (y \to z) \\
&\overset{(4.9)}{=} y \to (x \to z)\,.
\end{aligned}
$$

This completes the proof of the lemma. □

We have almost finished Theorem 1.3 thanks to Lemmas 4.1 and 4.2. All that remains is to check the independence of (L1) and (L2). As before, we just give the models.

On a 2-element set $\{0, 1\}$, define $x \to 0 = 0$ and $x \to 1 = 1$ for all x. This model satisfies (L1), but not (L2).

On a 2-element set $\{0, 1\}$, define $x \to y = 1$ for all x, y. This model satisfies (L2), but not (L1).

5. Problems

We start with an obvious question.

Problem 5.1. *Is the variety of MV-algebras* 1*-based?*

Obviously the axiom (M2) is rather long and involves four variables. This suggests the following.

Problem 5.2. *(1) Is there a* 2*-base for MV-algebras with at most three variables?*
(2) Is there a 2*-base for MV-algebras with one axiom no longer than (M1) and the other shorter than (M2)?*

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References

[1] J. Araújo and W. McCune, Computer solutions of problems in inverse semigroups, *Comm. Algebra* **38** (2010), 1104–1121.

[2] G. Cattaneo and F. Lombardo, Independent axiomatization of MV-algebras, Quantum structures, II (Liptovský Ján, 1998). *Tatra Mt. Math. Publ.* **15** (1998), 227–232.

[3] R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici, *Algebraic Foundations of Many-valued Reasoning*, Kluwer, Dordrecht-Boston-London, 2000.

[4] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.10*; 2007, (`http://www.gap-system.org`)

[5] A. Di Nola and A. Lettieri, Equational characterization of all varieties of MV-algebras, *J. Algebra* **221** (1999), 463–474.

[6] A. Dvurečenskij and T. Vetterlein, Algebras in the positive cone of *po*-groups, *Order* **19** (2002), 127–146.

[7] W. McCune, *Prover9 and Mace4*, version 2009-11A, (`http://www.cs.unm.edu/~mccune/prover9/`)

[8] D. Mundici, MV-algebras are categorically equivalent to bounded commutative BCK-algebras, *Math. Japon.* **31** (1986), 889–894.

[9] R. Padmanabhan and S. Rudeanu, Equational spectrum of Hilbert varieties, *Cent. Eur. J. Math.* **7** (2009), 66–72.

[10] M. Pałasiński and B. Woźniakowska, An equational basis for commutative BCK-algebras, *Math. Sem. Notes Kobe Univ.* **10** (1982), 175-178.
[11] H. Yutani, The class of commutative BCK-algebras is equationally definable, *Math. Sem. Notes Kobe Univ.* **5** (1977), 207–210.

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